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1978 J. Phys. A: Math. Gen. 11 843

(http://iopscience.iop.org/0305-4470/11/5/015)

J. Phys. A: Math. Gen., Vol. 11, No. 5, 1978. Printed in Great Britain

Exact invariants for time-dependent Hamiltonian systems with one degree-of-freedom

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Received 9 August 1977, in final form 30 December 1977

Abstract. Generalising the ideas of two previous papers a method is devised for obtaining exact invariants for time-dependent Hamiltonian systems with one degree-of-freedom. It consists in firstly transforming to a new Hamiltonian which is linear in the momentum variable, and secondly in solving the related Hamilton–Jacobi equation. The Hamiltonian of an oscillator with a supplementary inverse quadratic potential is treated as an illustrative example. After that, a complete application is given to a class of polynomial Hamiltonians, including an interpretation and discussion of the possible extent of the results.

1. Introduction

In the last ten years, there has been a growing interest in finding exact invariants for time-dependent Hamiltonian systems. Apart from their obvious importance in obtaining eventually a complete solution to the problem, such invariants appeared to be useful in different circumstances. For slowly time-varying oscillatory systems, the knowledge of an exact invariant resulted in an easy way of calculating an adiabatic invariant to all orders (Lewis 1968); a possibility for comparing exact and adiabatic invariants (Symon 1970), and a method for calculating the characteristic exponents of the Hill equation (Guyard *et al* 1971). The same type of invariants was used in quantum mechanics to solve the Schrödinger equation (Lewis and Riesenfeld 1969), and to discuss a relationship with the propagator in Feynman's path integral formulation (Khandekar and Lawande 1975). Recently, it was shown by Günther and Leach (1977) for a three-dimensional oscillator, and by Leach (1977) for a general n-dimensional quadratic Hamiltonian, that this invariant can take over the central role, played by the Hamiltonian for autonomous systems, to construct an invariant tensor, which in a natural way leads to a non-invariance dynamical symmetry group.

The starting point of the present investigation is linked to the way in which Lewis (1968) obtained his exact invariant for the time-dependent harmonic oscillator, namely the application in closed form of Kruskal's perturbation method (Kruskal 1962). In the first place we will focus attention on the system of partial differential equations for the determination of averaged variables. We will analyse a certain decoupling of the characteristic equations of their homogeneous part and derive corresponding necessary and sufficient conditions for the Hamiltonian of the system. Next, Hamiltonians satisfying these conditions will be shown to be characterised alternatively by the property that a canonical transformation exists, reducing them to

0305-4770/78/0005-0843$02.00

a form linear in the new momentum variable. This will lead us to the concept of linearisable Hamiltonians and a general method to calculate a related exact invariant. The method is first illustrated for the example of an oscillator with an inverse quadratic potential and then fully applied on a class of polynomial Hamiltonians.

2. Statement of the problem

Consider the first-order system of differential equations

$$\frac{\mathrm{d}q}{\mathrm{d}s}=\frac{\partial H}{\partial p}(q,p,t), \qquad \frac{\mathrm{d}p}{\mathrm{d}s}=-\frac{\partial H}{\partial q}(q,p,t), \qquad \frac{\mathrm{d}t}{\mathrm{d}s}=\epsilon, \tag{1}$$

with ϵ small, and under the assumption that a continuum of periodic solutions exists for $\epsilon=0$. This is the general form of a Hamiltonian system with one degree-of-freedom, depending on t through slowly varying parameters, and for which Kruskal's averaging method can be used to calculate successively an adiabatic invariant to all orders. In applying this method it is convenient to introduce first an intermediate set of variables,

$$y=f(H(q,p,t),t), \qquad \nu=\nu(q,p,t), \tag{2}$$

where ν is an angle-like variable. The new equations then have the form

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}s}&=\epsilon\frac{\partial y}{\partial t}=\epsilon g(y,\nu,t),\\ \frac{\mathrm{d}\nu}{\mathrm{d}s}&=[\nu,H]+\epsilon\frac{\partial\nu}{\partial t}=\psi(y,\nu,t),\end{aligned} \tag{3}$$

where $[\,.\,,\,.\,]$ is the Poisson bracket, and g and ψ are periodic functions of ν. The special form of the transformation formula of y is of course precisely chosen to make $\mathrm{d}y/\mathrm{d}s$ of order ϵ. Following the general idea of any averaging method one then tries to find a transformation to new variables z, ϕ ('nice variables' in Kruskal's terminology) such that the right-hand sides of the new equations do not depend on the angle variable ϕ. This means that the functions $z(y,\nu,t)$ and $\phi(y,\nu,t)$ will have to satisfy an equation of the type

$$\epsilon\frac{\partial F}{\partial y}g(y,\nu,t)+\frac{\partial F}{\partial\nu}\psi(y,\nu,t)+\epsilon\frac{\partial F}{\partial t}=G(z,t), \tag{4}$$

where G eventually can be chosen as simple as possible. Kruskal's method provides an iterative technique to solve this problem using series expansions in powers of the small parameter ϵ for all quantities involved. Moreover, using the expansions of z and ϕ, an adiabatic invariant can be calculated to all orders from the action integral over a ring of constant z,

$$J=\oint p\frac{\partial q}{\partial\phi}\,\mathrm{d}\phi. \tag{5}$$

It is clear that the method will be applicable in closed form as soon as the partial differential equations of type (4) are solvable in closed form. An important simplification towards the achievement of this objective is obtained when the characteristic equations of the homogeneous part of (4) are uncoupled, or equivalently when system (3) is uncoupled. Now a periodic dependence of g and ψ on the angle variable ν is inherent in the method. So a non-trivial uncoupling takes place when ψ is a function of ν and t only,

$$\psi = \psi(\nu, t). \tag{6}$$

Our first problem therefore will consist in identifying necessary and sufficient conditions for a Hamiltonian $H(t, q, p)$ in order to allow the construction of variables (y, ν) of the form (2) such that ψ as defined in (3) has the property (6). For Hamiltonians of the resulting class there is a good chance to solve equations (4) in Kruskal's method. However, we will look for a more direct method to obtain a partial differential equation of type (4) which enables the calculation of an exact invariant.

Remark. In the case of the harmonic oscillator (Lewis 1968) system (3) did not only have property (6), but also the property

$$g(y, \nu, t) = g_1(y) g_2(\nu, t), \tag{7}$$

which resulted in a solution of equations (4) by separation of variables. This feature was fully explored in two of our previous papers (Sarlet 1975a, b), but the corresponding class of Hamiltonians appears to be too restrictive now.

3. General theory

The system (3) will have property (6) if and only if a functional dependence comes about between $\mathrm{d}\nu/\mathrm{d}s$ and ν, with t as parameter. This is equivalent to the vanishing of a Jacobian, which in Poisson bracket notation (both concepts coincide for one degree-of-freedom) yields,

$$[[H, \nu], \nu] + \epsilon\left[\frac{\partial \nu}{\partial t}, \nu\right] = 0. \tag{8}$$

If $\nu(q, p, t)$ is required to be independent of ϵ, both terms must vanish separately. We then have†,

$$\left[\frac{\partial \nu}{\partial t}, \nu\right] = 0 \quad \Leftrightarrow \quad \frac{\partial \nu}{\partial q} = \chi(q, p)\frac{\partial \nu}{\partial p} \tag{9}$$

for some function χ, and with the help of this χ,

$$[[H, \nu], \nu] = 0 \quad \Leftrightarrow \quad \frac{\partial^2 H}{\partial q^2} - 2\chi\frac{\partial^2 H}{\partial q\, \partial p} + \chi^2\frac{\partial^2 H}{\partial p^2} + \frac{\partial \chi}{\partial p}\cdot\frac{\partial H}{\partial q} - \frac{\partial \chi}{\partial q}\cdot\frac{\partial H}{\partial p} = 0. \tag{10}$$

So we obtain the following.

Proposition 1. A Hamiltonian $H(q, p, t)$ allows a transition to new coordinates (y, ν) of the form (2), such that the new equations have property (6), if and only if there exists a function $\chi(q, p)$ satisfying equation (10).

Next we introduce the following concept.

† If the case $\partial\nu/\partial p = 0$, which will fit in with the theory later on, is excluded, then the if and only if condition is true.

Definition. A Hamiltonian $H(q, p, t)$ is linearisable if a time-independent canonical transformation exists, reducing it to a linear function of the new momentum variable.

An alternative characterisation of the Hamiltonians in proposition 1 is then provided by the following proposition.

Proposition 2. The class of Hamiltonians determined by proposition 1 coincides with the class of non-trivially† linearisable Hamiltonians.

Proof. A Hamiltonian is linearisable if and only if after a time-independent canonical transformation

$$(q, p) \leftrightarrow (Q, P)$$

the equation for Q becomes of the form

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \phi_1(Q, t),$$

which is equivalent to

$$\left[\frac{\mathrm{d}Q}{\mathrm{d}t}, Q\right] = 0, \qquad \text{or} \qquad [[H, Q], Q] = 0. \tag{11}$$

If such a reduction is possible with $\partial Q/\partial p = 0$, it means that the original Hamiltonian already was linear in p and thus is trivially linearisable. Excluding this case, we can put

$$\frac{\partial Q}{\partial q} = \chi(q, p)\frac{\partial Q}{\partial p}, \tag{12}$$

so that equation (11) is equivalent to equation (10). This completes the proof.

Now, in applying Kruskal's method an exact invariant should follow from the solution of the linear partial differential equations (4). But for the class of linearisable Hamiltonians, a linear partial differential equation leading to invariants of the system is immediately at hand, namely the Hamilton–Jacobi equation corresponding to the new Hamiltonian. Therefore, instead of using Kruskal's method we propose a calculation along the following lines. Given a Hamiltonian $H(q, p, t)$:

(i) Investigate whether or not H is linearisable. This requires the determination of a particular solution χ of equation (10)‡.
(ii) With this $\chi(q, p)$, find a particular solution $Q(q, p)$ of equation (12).
(iii) Determine a function $P(q, p)$, satisfying

$$[Q, P] = 1. \tag{13}$$

Steps (ii) and (iii) provide us with a canonical transformation reducing the Hamiltonian to the form

$$K(Q, P, t) = \phi_1(Q, t)P + \phi_2(Q, t). \tag{14}$$

† The non-triviality here means that the linearisation cannot be performed by the identity transformation.
‡ Since H depends on t and χ does not, this will often lead to a splitting up of equation (10) into several parts, which are easy to handle.

(iv) Determine a complete integral of the Hamilton–Jacobi equation

$$\phi_1(Q,t)\frac{\partial W}{\partial Q}+\phi_2(Q,t)+\frac{\partial W}{\partial t}=0, \tag{15}$$

with the property of being linear in the arbitrary constant I. That such an integral exists can be sketched as follows. Consider the characteristic equations of (15),

$$\frac{\mathrm{d}Q}{\phi_1(Q,t)}=\frac{\mathrm{d}t}{1}=\frac{\mathrm{d}W}{-\phi_2(Q,t)}. \tag{16}$$

Let

$$f_1(Q,t)=C_1 \qquad \text{or} \qquad Q=Q_1(C_1,t)$$

be the general solution of the first equation in (16) (C_1 being an arbitrary constant). Then the second equation can be written as

$$\frac{\mathrm{d}W}{\mathrm{d}t}=-\phi_2(Q_1(C_1,t),t).$$

If its general solution is denoted by

$$W=f_2(C_1,t)+C_2,$$

a complete integral of (15) is given by

$$W(Q,I,t)=f_2(f_1(Q,t),t)+If_3(f_1(Q,t)), \tag{17}$$

where I is the arbitrary constant and f_3 is an arbitrary function of its argument.

(v) Calculate I from the transformation formula $P=\partial W/\partial Q$, and express it in terms of the original variables.

Remark. The Hamilton–Jacobi equation associated with the given Hamiltonian can be highly non-linear. If a suitable function χ can be found in the above method, the solution of that non-linear equation is in fact replaced by the successive determination of a particular solution of a system of linear partial differential equations. This system consists of equation (10) (which is more precisely a quasi-linear one), equations (12), (13), and finally (15). Note that the calculation of the invariant $I(q,p,t)$ from (17) is again a linear process.

4. Examples

4.1. An oscillator with inverse quadratic potential

Consider the Hamiltonian

$$H=\tfrac{1}{2}p^2+\tfrac{1}{2}\omega^2(t)q^2+kq^{-2}. \tag{18}$$

Equation (10) becomes

$$\omega^2(t)+6kq^{-4}+\chi^2+\frac{\partial\chi}{\partial p}(\omega^2(t)q-2kq^{-3})-\frac{\partial\chi}{\partial q}p=0.$$

χ having to be independent of t, the coefficient of $\omega^2(t)$ must vanish separately, giving

$$\chi = -pq^{-1} + f(q),$$

with a yet arbitrary function f.

The remaining part of the equation can only be satisfied with a real function f if k is negative.

Putting,

$$8k = -\alpha^2, \tag{19}$$

we find $f = \alpha q^{-2}$.

So, (18) is a linearisable Hamiltonian for negative values of k, and then χ is given by

$$\chi = -pq^{-1} + \alpha q^{-2}. \tag{20}$$

The determination of a canonical transformation satisfying (12) and (13) is very simple and yields

$$Q = -pq^{-1} + \tfrac{1}{2}\alpha q^{-2}, \qquad P = \tfrac{1}{2}q^2. \tag{21}$$

It transforms the Hamiltonian (18) to the form

$$K(Q, P, t) = P(Q^2 + \omega^2(t)) - \tfrac{1}{2}\alpha Q, \tag{22}$$

for which the associated Hamilton–Jacobi equation is

$$\frac{\partial W}{\partial Q}(Q^2 + \omega^2) - \frac{1}{2}\alpha Q + \frac{\partial W}{\partial t} = 0. \tag{23}$$

A complete integral of this equation, calculated by the method explained above, is given by

$$W(Q, I, t) = I\left(\tan^{-1}[\rho(\rho Q + \dot\rho)] - \int \rho^{-2}\,dt\right) + \frac{\alpha}{4}\ln[\rho^{-2} + (\rho Q + \dot\rho)^2], \tag{24}$$

where $\rho(t)$ is a particular solution of the equation

$$\ddot\rho + \omega^2(t)\rho = \rho^{-3}. \tag{25}$$

From the transformation formula $P = \partial W/\partial Q$, and expressed in terms of the original variables, I is easily found to be,

$$I = \tfrac{1}{2}[\rho^{-2}q^2 + (\rho p - \dot\rho q)^2 + 2k\rho^2 q^{-2}], \tag{26}$$

which coincides with the expression given by Khandekar and Lawande (1975). Note that it turns out to be irrelevant that the Hamiltonian (18) was only linearisable for negative k, since the expression (26) of course remains an exact invariant for positive k.

4.2. *Application to a class of polynomial Hamiltonians*

Consider the following class of Hamiltonians, quadratic in the momentum variable:

$$H = \tfrac{1}{2}a(t)q^k p^2 + b(t)q^l p + c(t)q^m. \tag{27}$$

These are polynomial functions of q and p if k, l and m are positive integers, but in fact those constants might so far take any real value.

4.2.1. Linearisable Hamiltonians of type (27). Equation (10) here becomes,

$$\begin{aligned}\frac{\partial\chi}{\partial p}(\tfrac{1}{2}akq^{k-1}p^2+lbq^{l-1}p+mcq^{m-1})-\frac{\partial\chi}{\partial q}(aq^kp+bq^l)\\ =-\tfrac{1}{2}ak(k-1)q^{k-2}p^2-l(l-1)bq^{l-2}p-m(m-1)cq^{m-2}\\ +2\chi(akq^{k-1}p+blq^{l-1})-aq^k\chi^2.\end{aligned}\tag{28}$$

We suppose the functions $a(t)$, $b(t)$, and $c(t)$ to be linearly independent, so that their coefficients must vanish separately. For $a(t)$ we get

$$\frac{1}{2}kqp^2\frac{\partial\chi}{\partial p}-q^2p\frac{\partial\chi}{\partial q}=-\frac{1}{2}k(k-1)p^2+2kqp\chi-q^2\chi^2.\tag{29}$$

One of the characteristic equations of (29) immediately yields the first integral

$$p^2q^k=\text{constant, say } C_1^2,\tag{30}$$

which permits writing the second characteristic equation as

$$\frac{\mathrm{d}\chi}{\mathrm{d}q}=\frac{1}{2}k(k-1)C_1q^{-\frac{1}{2}k-2}-2kq^{-1}\chi+C_1^{-1}q^{\frac{1}{2}k}\chi^2.\tag{31}$$

(31) is a Riccati equation. It can be transformed to a linear second-order equation (see e.g. Murphy 1960) by the substitution

$$\chi=-C_1q^{\frac{1}{2}k}u^{-1}u' \qquad (u'=\mathrm{d}u/\mathrm{d}q).\tag{32}$$

This new equation, of the form

$$2q^ku''+3kq^{k-1}u'+k(k-1)q^{k-2}u=0$$

is easily solved, giving

$$u(q)=Bq^{-\frac{1}{2}k}+A(2-k)^{-1}q^{1-k},\tag{33}$$

where A and B are constants and we assume for the moment $k\neq 2$. Substituting (33) in (32), putting $C_2=BA^{-1}$, the general solution for χ is obtained when C_1 is replaced from (30) and C_2 is replaced by an arbitrary function f_1 of C_1. So,

$$\chi_1=\tfrac{1}{2}kpq^{-1}-\tfrac{1}{2}p[f_1(p^2q^k)q^{\frac{1}{2}k}+(2-k)^{-1}q]^{-1}.\tag{34}$$

The coefficient of $b(t)$ in (28) will disappear if

$$l\frac{\partial\chi}{\partial p}qp-q^2\frac{\partial\chi}{\partial q}=-l(l-1)p+2lq\chi,\tag{35}$$

that is for,

$$\chi_2=lpq^{-1}+f_2(pq^l)q^{-2l}.\tag{36}$$

The coefficient of $c(t)$ requires χ to satisfy

$$m\frac{\partial\chi}{\partial p}q^{m-1}=-m(m-1)q^{m-2}.\tag{37}$$

We omit the solution $m=0$, since an additive function of time is unimportant in the Hamiltonian (27). The solution of (37) then is,

$$\chi_3 = -(m-1)pq^{-1} + f_3(q), \tag{38}$$

(f_2 and f_3 again are arbitrary functions).

It is easily seen that a common expression χ in equations (34), (36), (38) can only be of the form Cpq^{-1} for some constant C. An expression of that type is obtained from (34) in two cases:

(i) $\quad f_1 \equiv 0, \quad$ then $\chi = (k-1)pq^{-1}$, (39)

(ii) $\quad f_1^{-1} \equiv 0, \quad$ then $\chi = \frac{1}{2}kpq^{-1}$. (40)

We keep k arbitrary. In the first case χ_2 will coincide with (39) if we choose $f_2 = 0$ and $l = k-1$, or $l = 1$ and $f_2 = (k-2)pq$. χ_3 will coincide with the same expression for the choice $f_3 = 0$, $m = 2-k$.

In the second case, to obtain (40) out of χ_2 and χ_3 we have to choose $f_2 = 0$ and $l = \frac{1}{2}k$, or $l = 1$ and $f_2 = (\frac{1}{2}k - 1)pq$, together with $f_3 = 0$ and $m = 1 - \frac{1}{2}k$.

Hence altogether we have four classes of linearisable Hamiltonians of the form (27), namely

$$H_1 = \tfrac{1}{2}a(t)q^k p^2 + b(t)q^{k-1}p + c(t)q^{2-k}, \tag{41}$$

$$H_2 = \tfrac{1}{2}a(t)q^k p^2 + b(t)qp + c(t)q^{2-k}, \tag{42}$$

$$H_3 = \tfrac{1}{2}a(t)q^k p^2 + b(t)q^{\frac{1}{2}k}p + c(t)q^{1-\frac{1}{2}k}, \tag{43}$$

$$H_4 = \tfrac{1}{2}a(t)q^k p^2 + b(t)qp + c(t)q^{1-\frac{1}{2}k}, \tag{44}$$

H_1 and H_2 correspond to (39) as solution for χ, H_3 and H_4 correspond to (40). The solution one obtains through a separate calculation for the previously excluded case $k=2$ is precisely the common form to which all Hamiltonians H_i ($i = 1, \ldots, 4$) reduce for $k=2$. (Note further that the Hamiltonians H_i of course remain linearisable when the time-dependent coefficients are no longer linearly independent).

4.2.2. Transformation to Hamiltonians linear in the momentum variable. For χ given by (39), the canonical transformation (we again have to exclude for the moment the case $k=2$)

$$Q = pq^{k-1}, \qquad P = (k-2)^{-1}q^{2-k} \tag{45}$$

reduces the Hamiltonians H_1 and H_2 to the following linear expressions in P:

$$K_1 = (k-2)P(\tfrac{1}{2}a(t)Q^2 + c(t)) + b(t)Q, \tag{46}$$

$$K_2 = (k-2)P(\tfrac{1}{2}a(t)Q^2 + b(t)Q + c(t)). \tag{47}$$

For χ given by (40), a canonical transformation satisfying equations (12) and (13) is,

$$Q = pq^{\frac{1}{2}k}, \qquad P = \left(\frac{k}{2} - 1\right)^{-1} q^{1-\frac{1}{2}k}. \tag{48}$$

It transforms H_3 and H_4 to

$$K_3 = \left(\frac{k}{2} - 1\right)c(t)P + \tfrac{1}{2}a(t)Q^2 + b(t)Q, \tag{49}$$

$$K_4 = \left(\frac{k}{2} - 1\right) P(b(t)Q + c(t)) + \tfrac{1}{2}a(t)Q^2. \tag{50}$$

4.2.3. Exact invariants for the Hamiltonians H_i. The Hamilton–Jacobi equation for K_1 becomes

$$\frac{\partial W}{\partial Q}(a_1(t)Q^2 + c_1(t)) + b(t)Q + \frac{\partial W}{\partial t} = 0, \tag{51}$$

where for the sake of simplicity we have put

$$a_1(t) = \tfrac{1}{2}(k-2)a(t), \qquad c_1(t) = (k-2)c(t). \tag{52}$$

The first characteristic equation of (51),

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = a_1 Q^2 + c_1$$

has the general solution,

$$\tan^{-1}[\rho^2(Q + a_1^{-1}\rho^{-1}\dot{\rho})] - \int a_1 \rho^{-2}\,\mathrm{d}t = C_1, \tag{53}$$

where C_1 is an arbitrary constant and $\rho(t)$ is any particular solution of the equation

$$\ddot{\rho} - a_1^{-1}\dot{a}_1\dot{\rho} + \rho a_1 c_1 = a_1^2 \rho^{-3}. \tag{54}$$

With the help of (53), the second characteristic equation, $\mathrm{d}W/\mathrm{d}t = -b(t)Q$, can be written in the form,

$$\frac{\mathrm{d}W}{\mathrm{d}t} = b a_1^{-1}\rho^{-1}\dot{\rho} - b\rho^{-2}\tan\left(C_1 + \int a_1\rho^{-2}\,\mathrm{d}t\right). \tag{55}$$

Without specifying the nature of the functions $a_1(t)$ and $b(t)$, it is not possible to give an analytic expression for the general solution of equation (55), except in the special case that $b(t)$ is proportional to $a_1(t)$. Indeed, putting

$$b a_1^{-1} = \kappa_1, \text{ a constant}, \tag{56}$$

and

$$\nu = C_1 + \int a_1 \rho^{-2}\,\mathrm{d}t, \tag{57}$$

equation (55) becomes

$$\mathrm{d}W = \kappa_1 \mathrm{d}\ln\rho - \kappa_1 \tan\nu\,\mathrm{d}\nu.$$

The integration is straightforward; in accordance with the ideas explained in § 3 a complete integral of equation (51) is given by

$$W(Q, I, t) = -\tfrac{1}{2}\kappa_1 \ln[\rho^{-2} + \rho^2(Q + a_1^{-1}\rho^{-1}\dot{\rho})^2]$$
$$+ I\left(\tan^{-1}[\rho^2(Q + a_1^{-1}\rho^{-1}\dot{\rho})] - \int a_1\rho^{-2}\,\mathrm{d}t\right). \tag{58}$$

Solving the transformation formula $P = \partial W/\partial Q$ for I, we obtain

$$I = P[\rho^{-2} + \rho^2(Q + a_1^{-1}\rho^{-1}\dot{\rho})^2] + \kappa_1\rho^2(Q + a_1^{-1}\rho^{-1}\dot{\rho}). \tag{59}$$

Finally, if we put $(k-2)\kappa_1 = 2\kappa$, which implies

$$\kappa = ba^{-1}, \tag{60}$$

$(k-2)I = I_1$, and make use of the transformation formulae (45), we get the following expression for an exact invariant I_1 of the original Hamiltonian H_1 (under the restriction (60)):

$$I_1^{(1)} = \rho^{-2}q^{2-k} + q^k(\rho p + a_1^{-1}\dot{\rho}q^{1-k})^2 + 2\kappa\rho q^{k-1}(\rho p + a_1^{-1}\dot{\rho}q^{1-k}), \tag{61}$$

ρ still being any particular solution of equation (54) (the superscript (1) is added to relate I_1 to the Hamiltonian H_1).

For K_2 we have to solve the Hamilton–Jacobi equation

$$\frac{\partial W}{\partial Q}(a_1Q^2 + 2b_1Q + c_1) + \frac{\partial W}{\partial t} = 0, \tag{62}$$

where $a_1(t) = \frac{1}{2}(k-2)a(t)$, $b_1(t) = \frac{1}{2}(k-2)b(t)$, $c_1(t) = (k-2)c(t)$. A complete integral is given by

$$W = I\left(\tan^{-1}[\rho^2(Q + a_1^{-1}(b_1 + \rho^{-1}\dot{\rho}))] - \int a_1\rho^{-2}\,\mathrm{d}t\right), \tag{63}$$

where $\rho(t)$ is a particular solution of

$$\ddot{\rho} - a_1^{-1}\dot{a}_1\dot{\rho} + \rho(a_1c_1 - b_1^2 - a_1^{-1}\dot{a}_1b_1 + \dot{b}_1) = a_1^2\rho^3. \tag{64}$$

Putting $(k-2)I = I_1$, we get the following invariant, expressed in terms of the original variables:

$$I_1^{(2)} = \rho^{-2}q^{2-k} + \rho^2q^{2-k}[pq^{k-1} + a_1^{-1}(b_1 + \rho^{-1}\dot{\rho})]^2. \tag{65}$$

The structure of classes K_3 and K_4 is essentially different from that of K_1 and K_2, the difference being that the coefficient of P does not contain a term in Q^2. This simplifies the problem very much. Indeed, one can simply premise an invariant I linear in P and Q.

This time making use of the transformation formulae (48) and putting $I_2 = (\frac{1}{2}k-1)I$, $f_2 = (\frac{1}{2}k-1)f$, $g_2 = (\frac{1}{2}k-1)g$, $a_2 = (\frac{1}{2}k-1)a$, $b_2 = (\frac{1}{2}k-1)b$, $c_2 = (\frac{1}{2}k-1)c$, we get as invariant for the Hamiltonian H_3,

$$I_2^{(3)} = q^{1-\frac{1}{2}k} + f_2pq^{\frac{1}{2}k} + g_2, \tag{66}$$

with

$$f_2 = \int a_2\,\mathrm{d}t, \qquad g_2 = \int (b_2 - f_2c_2)\,\mathrm{d}t, \tag{67}$$

while the resulting invariant for H_4 is,

$$I_2^{(4)} = \alpha_2(t)q^{1-\frac{1}{2}k} + \beta_2(t)pq^{\frac{1}{2}k} + \gamma_2(t), \tag{68}$$

with

$$\alpha_2 = \exp\left(\int b_2\,\mathrm{d}t\right), \qquad \beta_2 = \alpha^{-1}\int \alpha^2a_2\,\mathrm{d}t, \qquad \gamma_2 = -\int \beta_2c_2\,\mathrm{d}t. \tag{69}$$

For the sake of completeness there remains one case to be treated, namely the case $k = 2$. The Hamiltonians H_i $(i = 1, \ldots, 4)$ then all reduce to the form

$$H = \tfrac{1}{2}a(t)q^2p^2 + b(t)qp + c(t). \tag{70}$$

A solution for χ is,

$$\chi = pq^{-1}, \tag{71}$$

and the linearising canonical transformation (from equations (12) and (13)) is given by

$$Q = qp, \qquad P = \ln p. \tag{72}$$

If we omit the additive function $c(t)$ in (70), the new Hamiltonian becomes

$$K = \tfrac{1}{2}a(t)Q^2 + b(t)Q. \tag{73}$$

This is of course a very special case of a Hamiltonian linear in the momentum variable P, since it is independent of P. Q is an invariant and the equations of motion for (73) can be trivially solved.

So, apart from the supplementary restriction (60) for H_1, we have obtained an exact invariant for all our linearisable Hamiltonians of type (27).

5. Discussion

In this paper, generalising some of the ideas of two previous articles, we have proposed a general procedure for solving time-dependent Hamiltonian systems with one degree-of-freedom (or for calculating an exact invariant of the system). The method is applicable to a special class of Hamiltonians, called linearisable Hamiltonians, and consists in transforming the system first (by a time-independent transformation) to a new one with linear Hamiltonian with respect to P, and secondly in solving the linear Hamilton–Jacobi equation. Recently another systematic method for the same problem was derived by Leach (1977). It is valid for n-dimensional Hamiltonian systems but mainly restricted to Hamiltonians which are quadratic in all variables. In this method the Hamiltonian is reduced to a similar form with constant coefficients by a linear time-dependent canonical transformation. In order to establish a relationship between both methods for the application discussed in the previous section, we make the following consideration. The canonical transformation,

$$P = pq^{\frac{1}{2}k}, \qquad Q = (1 - \tfrac{1}{2}k)^{-1} q^{1-\frac{1}{2}k}, \tag{74}$$

reduces the Hamiltonians H_i ($i = 1, \ldots, 4$) to the form,

$$\mathcal{H}_1 = \tfrac{1}{2}a(t)P^2 + b(t)PQ^{-1} + c(t)Q^2, \tag{75}$$

$$\mathcal{H}_2 = \tfrac{1}{2}a(t)P^2 + b(t)QP + c(t)Q^2, \tag{76}$$

$$\mathcal{H}_3 = \tfrac{1}{2}a(t)P^2 + b(t)P + c(t)Q, \tag{77}$$

$$\mathcal{H}_4 = \tfrac{1}{2}a(t)P^2 + b(t)QP + c(t)Q. \tag{78}$$

Apart from the first one, these Hamiltonians are quadratic in both Q and P. This means that the invariants (65), (66) and (68) could as well be obtained using Leach's method for the Hamiltonians (76), (77), (78) and transforming the results with the aid of (74). The Hamiltonian $\mathcal{H}_1$, which is the exception for the above relationship, has a different interesting feature. In solving the Hamilton–Jacobi equation (in the case $ba_1^{-1} = \kappa_1$) for the related Hamiltonian K_1, we found a complete integral (58), which is very similar in structure to the expression (24), obtained for the example in § 4.1. There is a perhaps rather surprising explanation for this resemblance. From the

Hamiltonian (18), assuming again that k is negative so that for example (19) holds, we get the following second-order equation of motion in q:

$$\ddot{q}+\omega^2(t)q=-\tfrac{1}{4}\alpha^2q^{-3}. \tag{79}$$

Now, putting in $\mathscr{H}_1$, $a(t)=1$, $b(t)=\frac{1}{2}\alpha$, and $c(t)=\frac{1}{2}\omega^2(t)$, we get exactly the same second-order equation for Q, which means that (18) and (75) then are so called 'q-equivalent Hamiltonians'. This concept was discussed in various papers, mainly by Havas (1957) and Currie and Saletan (1966). Of course, the formulae for the invariants I of both Hamiltonians are different, expressed as they are in terms of q and p. They would become identical when expressed in terms of q and $\dot{q}$. When a given Hamiltonian is not linearisable, it is an open question whether one could get around the difficulty by passing to a q-equivalent Hamiltonian.

Finally, something must be said about the nature of the second-order equation for q resulting from Hamiltonians of type (27). This equation of motion (for constant a) is of the form,

$$\ddot{q}-\tfrac{1}{2}kq^{-1}\dot{q}^2+f(q,t)=0, \tag{80}$$

where $f(q,t)$ depends on the choice of the Hamiltonians in the classes H_i $(i=1,\ldots,4)$.

Equations containing a quadratic term in $\dot{q}$ naturally arise, e.g. when a two-dimensional system is considered, consisting of a single particle which is constrained to move on a prescribed smooth curve (for an example see Andronov *et al* 1966). The time dependence of $f(q,t)$ in (80) then can come from slowly varying parameters. Let us give one example here: the motion of a particle moving on a branch of the cusp

$$x^{2/3}+y^{2/3}=d^{2/3},$$

can be described by a Hamiltonian of type (27), with $k=2/3$.

Acknowledgment

We would like to thank Professor R Mertens for his continuing interest in our work.

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